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COUPLED FREQUENCIES OF A HYDROELASTIC VISCOUS LIQUID SYSTEM

HELMUT F. BAUER
Universität der Bundeswehr München, Institut für Raumfahrttechnik, München, F.R.G.

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Abstract—The coupled frequencies of a hydroelastic system consisting of an elastic shell and a viscous liquid layer with a free surface have been treated. The system exhibits no z-dependency and may be either an annular liquid layer around an elastic center shell or a liquid layer inside an elastic container. The first case has been evaluated numerically, where the influence of the liquid surface tension parameter, the elasticity parameter of the shell and the thickness of the layer have been determined. In contrast to the hydroelastic system with an ideal liquid, the system with viscous liquid exhibits instability of the liquid surface as well as the shell.

NOTATION

a	radius of liquid, or radius of container shell
b	radius of shell, or radius of liquid surface
$\bar{D}$	$Eh/(1-\bar{\nu}^2)$
E	modulus of elasticity
h	shell wall thickness
I_m, K_m	modified Bessel functions of the first and second kind
k	diameter ratio, b/a
p	liquid pressure
r, ϕ	polar coordinates
s	complex frequency, $\bar{\sigma}+i\bar{\omega}$
t	time
T	surface tension of liquid
u	radial velocity component of liquid
v	angular velocity component of liquid
μ	dynamic viscosity of liquid
ξ	radial shell deflection
η	angular shell deflection
$\bar{\nu}$	Poisson ratio
ν	kinematic viscosity of liquid, μ/ρ
ρ	density of liquid
$\bar{\rho}$	density of shell material
Ψ	stream function
ζ	free liquid surface elevation
σ_r, $\tau_{r\phi}$	normal and shear stress

1. INTRODUCTION

The availability of extended manned space flights in an earth-orbiting laboratory make unique exploratory experiments possible. Such experiments can hardly be achieved under the action of gravity on earth. One of these experiments is the behavior of an annular liquid layer around a cylindrical center shell which on earth would, due to the hydrostatic pressure, bulge out to a geometric configuration not representing in the undisturbed equilibrium position a circular annular liquid cylinder. In zero-gravity (or micro-gravity) conditions the liquid is, however, held only by surface tension and due to the lack of gravity, its cylindrical form offers a number of advantages for exploratory experiments. The natural frequencies of such liquid layers, thick or extremely thin, should be known, in order to avoid certain frequency ranges during the experiments appearing in a space Laboratory, known as the so-called g-jitter, that may be occurring due to the operation of machines on board, the motion of the crew, etc. For a rigid center shell the natural frequencies of a liquid layer around it have been given for frictionless[1] and viscous[2] liquids, where the three-dimensional viscous case has not been evaluated numerically. For a system with no z-dependency as is being treated here (two-dimensional case), the natural frequencies of frictionless and viscous liquids have been presented[3]; numerical results are also exhibited. The condition of a rigid cylindrical center shell may, however, be violated, such that the shell may be elastic and perform vibrations. Also of interest is the case of an annular liquid layer inside a cylindrical container, capable of structural vibrations, which will excite the motion of the free liquid surface. It is of interest in such cases to know the shifting of the natural frequencies of the structure and liquid. We are therefore interested in the behavior of the coupled liquid–structure system as a function of the thickness of the liquid layer, its surface tension, and viscosity and of the elasticity of the shell, its thickness and the density ratio of the structure and liquid. Lord Rayleigh[4] treated the three-dimensional case of a cylindrical air column in an infinite medium and found, that the axisymmetric cylindrical configuration becomes unstable and exhibits its most pronounced instability at an axial wavelength $\lambda_z = 12.96a$. Lamb[5] performed a similar investigation, where mainly the simple liquid column was treated. In addition, the remarkable work of Plateau[6], who performed a large number of useful simulation experiments, should be mentioned. No work, however, has been performed yet upon the interaction of an elastic structure with a liquid in a zero-gravity environment.

The following investigation therefore represents the analysis for the determination of the coupled frequencies of a two-dimensional liquid–structure system, i.e. a system exhibiting no z-dependency. Our main interest is directed towards the behavior of the coupled viscous liquid–shell system, where the influence of the tension parameter $Ta/\rho\nu^2$, the thickness of the liquid layer ($k = b/a$; diameter ratio of shell and free liquid surface location), the density ratio of shell and liquid $\bar{\rho}/\rho$, the thickness of the shell h and the elasticity parameter $Ea^2/(1-\bar{\nu}^2)\bar{\rho}\nu^2k^2$ will be investigated.

2. BASIC EQUATIONS AND SOLUTION

Under the influence of an elastic structure a viscous liquid in a micro-gravity environment will execute damped vibrations, if disturbed (Fig. 1). The problem at hand is therefore the determination of the coupled frequencies of the elastic structure and the viscous liquid with a free surface displacement. For small liquid velocity components and small elastic deflections the governing equations may be linearized. The motion of the viscous liquid shall be obtained from the Stokes equations

$$\frac{1}{\nu}\frac{\partial u}{\partial t} + \frac{1}{\mu}\frac{\partial p}{\partial r} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} - \frac{2}{r^2}\frac{\partial v}{\partial \phi} \tag{1}$$

$$\frac{1}{\nu}\frac{\partial v}{\partial t} + \frac{1}{\mu}\frac{1}{r}\frac{\partial p}{\partial \phi} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial u}{\partial \phi} \tag{2}$$

and the continuity equation

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi} = 0 \tag{3}$$

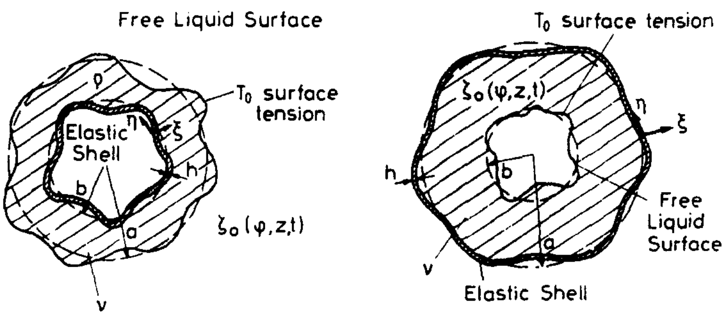

Fig. 1. Geometry and coordinate system.

where μ is the dynamic and $\nu = \mu/\rho$ the kinematic viscosity of the liquid and where $\mathbf{v} = u\mathbf{e}_r + v\mathbf{e}_\phi$ is the velocity of the liquid. These equations have to be solved with the appropriate boundary conditions. If the free liquid surface is at $r = a$ outside an elastic inner cylinder at $r = b$, $b < a$, the free surface condition is obtained with the kinematic condition at the free surface $\partial\zeta_0/\partial t = u$ from

$$\sigma_r + p_{T_0} = 0 \quad \text{and} \quad \tau_{r\phi} = 0 \qquad \text{at } r = a.$$

This renders

$$p - 2\mu\frac{\partial u}{\partial r} = \frac{T_0}{a} - \frac{T_0}{a^2}\left[\zeta_0 + \frac{\partial^2\zeta_0}{\partial\phi^2}\right] \qquad \text{at } r = a,$$

which, after differentiation with respect to the time t, yields the dynamic free surface condition

$$\frac{\partial p}{\partial t} - 2\mu\frac{\partial^2 u}{\partial r\,\partial t} + \frac{T_0}{a^2}\left[u + \frac{\partial^2 u}{\partial\phi^2}\right] = 0 \qquad \text{at } r = a, \tag{4}$$

while $\tau_{r\phi} = 0$ gives

$$r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial\phi} = 0 \qquad \text{at } r = a. \tag{5}$$

If we denote the elastic deflection of the structure with ξ in the radial and η in the circumferential direction, the boundary condition at the inner cylindrical structure is given by

$$\frac{\partial\xi}{\partial t} = u \quad \text{and} \quad \frac{\partial\eta}{\partial t} = v \qquad \text{at } r = b. \tag{6}$$

A cylindrical shell of infinite length renders under the assumption of no motion in the z-direction the equations[7]:

$$\frac{\partial\eta}{\partial\phi} + \xi + \frac{h^2}{12b^2}\left[\xi + 2\frac{\partial^2\xi}{\partial\phi^2} + \frac{\partial^4\xi}{\partial\phi^4}\right] + \frac{\bar{\rho}(1-\bar{\nu}^2)b^2}{E}\frac{\partial^2\xi}{\partial t^2} = -\left[p - 2\mu\frac{\partial u}{\partial r}\right]\frac{b^2}{\bar{D}} \qquad \text{at } r = b, \tag{7}$$

$$\frac{\partial^2\eta}{\partial\phi^2} + \frac{\partial\xi}{\partial\phi} - \frac{\bar{\rho}(1-\bar{\nu}^2)b^2}{E}\frac{\partial^2\eta}{\partial t^2} = \frac{b^2\mu}{\bar{D}}\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial\mu}{\partial\phi}\right] \qquad \text{at } r = b \tag{8}$$

where $\xi = \xi(\phi, t)$, $\eta = \eta(\phi, t)$, $\bar{\nu}$ is Poisson's ratio, E the elasticity modulus and $\bar{D} = Eh/(1-\bar{\nu}^2)$, with h as the thickness of the shell. The value $\bar{\rho}$ is the density of the elastic shell. As may be noticed, these equations are coupled with the liquid motion on the right-hand side. If the liquid is in an elastic container and exhibits an inner free liquid surface, then the free surface condition at $r = b$ reads:

$$\frac{\partial p}{\partial t} - 2\mu\frac{\partial^2 u}{\partial r\,\partial t} - \frac{T_i}{b^2}\left[u + \frac{\partial^2 u}{\partial\phi^2}\right] = 0 \qquad \text{at } r = b \tag{9}$$

and

$$r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial\phi} = 0 \qquad \text{at } r = b, \tag{10}$$

while the two-dimensional shell eqns are given by expressions (7) and (8), in which the sign of the term $-[p-2\mu(\partial u/\partial r)]a^2/\bar{D}$, has to be changed to a plus. In addition b has to be substituted by a. By introducing the stream function $\Psi(r,\phi,t)$, such that the continuity equation is satisfied identically, i.e.

$$u = -\frac{1}{r}\frac{\partial\Psi}{\partial\phi} \quad \text{and} \quad v = \frac{\partial\Psi}{\partial r}, \tag{11}$$

one obtains after eliminating the pressure from the Stokes' eqns (1) and (2), the partial differential equation

$$\Delta\left(\Delta\Psi - \frac{1}{\nu}\frac{\partial\Psi}{\partial t}\right) = 0 \tag{12}$$

where

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}.$$

By the application of the vector operation "divergence" to the Stokes' eqns, we obtain the Laplace eqn for the pressure distribution, i.e.

$$\Delta p = 0. \tag{13}$$

Assuming for all values (velocities u and v, pressure p and the deflections ξ and η), the dependency

$$e^{im\phi+st}$$

yields the differential equation for the stream function $\Psi^*(\Psi = \Psi^*(r)e^{im\phi+st})$

$$\bar{\Delta}\left[\bar{\Delta}\Psi^* - \frac{s}{\nu}\Psi^*\right] = 0 \tag{14}$$

where

$$\bar{\Delta} \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}.$$

The solution of this equation yields (summations in m have been omitted)

$$\Psi^*(r) = Ar^m + BI_m\left(\sqrt{\frac{s}{\nu}}r\right) + C/r^m + DK_m\left(\sqrt{\frac{s}{\nu}}r\right) \tag{15}$$

where I_m and K_m, are the modified Bessel functions of the first and second kind and of order m. With this result and eqn (11), the velocity distribution is given by

$$u(r,\phi,t) = -im\,e^{im\phi+st}\left[Ar^{m-1} + BI_m\left(\sqrt{\frac{s}{\nu}}r\right)\Big/r + C/r^{m+1} + DK_m\left(\sqrt{\frac{s}{\nu}}r\right)\Big/r\right] \tag{16}$$

and

$$v(r,\phi,t) = e^{im\phi+st}\left[Amr^{m-1} + B\sqrt{\frac{s}{\nu}}I'_m\left(\sqrt{\frac{s}{\nu}}r\right) - Cm/r^{m+1} + D\sqrt{\frac{s}{\nu}}K'_m\left(\sqrt{\frac{s}{\nu}}r\right)\right]. \tag{17}$$

In this eqn, the prime indicates differentiation with respect to the arguments. The pressure distribution as obtained from eqn (13) yields

$$p(r,\phi,t) = [Er^m + F/r^m]\,e^{im\phi+st}. \tag{18}$$

Introducing eqns (18) and (16) and (17) into the Stokes' eqn renders a relation of the constants E and F to A and C. It is

$$E = \rho isA \quad \text{and} \quad F = -\rho isC. \tag{19}$$

2.1. *Viscous liquid around elastic center shell*

If a viscous liquid is placed around an elastic center shell, we have to employ for the determination of the frequency equation eqns (4)–(8). The radial and circumferential deflections are given by

$$\xi = X e^{im\phi+st} \quad \text{and} \quad \eta = Y e^{im\phi+st}. \tag{20}$$

From eqns (4) and (5) we obtain

$$\rho s^2 Aa^{m+2} - \rho s^2 C/a^{m-2} + 2s\mu m\Big\{A(m-1)a^m + B\Big[a\sqrt{\tfrac{s}{\nu}}I_{m-1}\Big(\sqrt{\tfrac{s}{\nu}}a\Big) - (m+1)I_m\Big(\sqrt{\tfrac{s}{\nu}}a\Big)\Big]$$
$$- C(m+1)/a^m - D\Big[a\sqrt{\tfrac{s}{\nu}}K_{m-1}\Big(\sqrt{\tfrac{s}{\nu}}a\Big) + (m+1)K_m\Big(\sqrt{\tfrac{s}{\nu}}a\Big)\Big]\Big\}$$
$$+ \frac{T_0}{a}m(m^2-1)\Big[Aa^m + BI_m\Big(\sqrt{\tfrac{s}{\nu}}a\Big) + C/a^m + DK_m\Big(\sqrt{\tfrac{s}{\nu}}a\Big)\Big] = 0 \tag{21}$$

and

$$2m(m-1)a^m A + B\Big\{\Big[2m(m+1) + \frac{sa^2}{\nu}\Big]I_m\Big(\sqrt{\tfrac{s}{\nu}}a\Big) - 2\sqrt{\tfrac{s}{\nu}}a\cdot I_{m-1}\Big(\sqrt{\tfrac{s}{\nu}}a\Big)\Big\}$$
$$+ 2m(m+1)C/a^m + D\Big\{\Big[2m(m+1) + \frac{sa^2}{\nu}\Big]\cdot K_m\Big(\sqrt{\tfrac{s}{\nu}}a\Big)$$
$$+ 2\sqrt{\tfrac{s}{\nu}}aK_{m-1}\Big(\sqrt{\tfrac{s}{\nu}}a\Big)\Big\} = 0. \tag{22}$$

Equations (6), with (16) and (17) and $k = b/a$ render

$$-im\Big[Ab^m + BI_m\Big(k\sqrt{\tfrac{s}{\nu}}a\Big) + \frac{C}{b^m} + DK_m\Big(k\sqrt{\tfrac{s}{\nu}}a\Big)\Big] = sXb \tag{23}$$

$$Amb^m + Ba\sqrt{\tfrac{s}{\nu}}kI'_m\Big(k\sqrt{\tfrac{s}{\nu}}a\Big) - \frac{Cm}{b^m} + Da\sqrt{\tfrac{s}{\nu}}k\cdot K'_m\Big(k\sqrt{\tfrac{s}{\nu}}a\Big) = sYb. \tag{24}$$

Introducing the above results, i.e. eqns (16)–(20), into the shell eqns (7) and (8), renders two more algebraic equations in the unknown constants A, B, C and D. We thus obtain:

$$2m(m-1)b^m A + B\Big\{\Big[2m(m+1) + \frac{sa^2}{\nu}k^2\Big]I_m\Big(k\sqrt{\tfrac{s}{\nu}}a\Big) - 2k\sqrt{\tfrac{s}{\nu}}a\cdot I_{m-1}\Big(k\sqrt{\tfrac{s}{\nu}}a\Big)\Big\}$$
$$+ 2m(m+1)\frac{C}{b^m} + D\Big\{\Big[2m(m+1) + \frac{sa^2}{\nu}k^2\Big]K_m\Big(k\sqrt{\tfrac{s}{\nu}}a\Big) + 2k\sqrt{\tfrac{s}{\nu}}aK_{m-1}\Big(k\sqrt{\tfrac{s}{\nu}}a\Big)\Big\}$$
$$- \frac{im\bar{D}}{\mu}X + \frac{\bar{D}}{\mu}\Big[m^2 + \frac{\bar{\rho}(1-\bar{\nu}^2)}{E}s^2b^2\Big]Y = 0 \tag{25}$$

and

$$Ab^m[2\mu(m-1)m+\rho sb^2]+2m\mu B\left[k\sqrt{\frac{s}{v}}aI'_m\left(k\sqrt{\frac{s}{v}}a\right)-I_m\left(k\sqrt{\frac{s}{v}}a\right)\right]$$

$$-\frac{C}{b^m}[2\mu m(m+1)+\rho sb^2]+2\mu mD\cdot\left[k\sqrt{\frac{s}{v}}aK'_m\left(k\sqrt{\frac{s}{v}}a\right)-K_m\left(k\sqrt{\frac{s}{v}}a\right)\right]$$

$$-\bar{D}\left[1+\frac{h^2}{12a^2k^2}(1-2m^2+m^4)+\frac{\bar{\rho}b^2s^2}{E}(1-\bar{v}^2)\right]iX+m\bar{D}Y=0. \quad (26)$$

Equations (21)–(26) represent six homogeneous algebraic equations, of which the vanishing of the coefficient determinant represents the frequency equation ($j,l=1,2\ldots6$)

$$\|\delta_{jl}\|=0. \quad (27)$$

The elements of this determinant are given with

$$x\equiv a\sqrt{\frac{s}{v}},\quad \alpha\equiv\frac{\bar{D}a}{\mu v},\quad \beta\equiv\frac{\bar{\rho}(1-\bar{v}^2)v^2}{Ea^2},\quad T_0^*\equiv\frac{T_0a}{\rho v^2}$$

by

$$\begin{aligned}
&\delta_{11}=x^4+2x^2m(m-1)+T_0^*m(m^2-1)\\
&\delta_{12}=2mx^2[xI_{m-1}(x)-(m+1)I_m(x)]+T_0^*m(m^2-1)I_m(x)\\
&\delta_{13}=T_0^*m(m^2-1)-[2m(m+1)x^2+x^4]\\
&\delta_{14}=-2mx^2[xK_{m-1}(x)+(m+1)K_m(x)]+T_0^*m(m^2-1)K_m(x)\\
&\delta_{15}=\delta_{16}=0\\
&\delta_{21}=2m(m-1),\quad \delta_{22}=[2m(m+1)+x^2]I_m(x)-2xI_{m-1}(x)\\
&\delta_{23}=2m(m+1),\quad \delta_{24}=[2m(m+1)+x^2]K_m(x)+2xK_{m-1}(x)\\
&\delta_{25}=\delta_{26}=0\\
&\delta_{31}=mk^m,\quad \delta_{32}=mI_m(kx),\quad \delta_{33}=m/k^m,\quad \delta_{34}=mK_m(kx)\\
&\delta_{35}=-ikx^2,\quad \delta_{36}=0\\
&\delta_{41}=mk^m,\quad \delta_{42}=kxI'_m(kx),\quad \delta_{43}=-m/k^m\\
&\delta_{44}=kxK'_m(kx),\quad \delta_{45}=0,\quad \delta_{46}=-kx^2\\
&\delta_{51}=2m(m-1)k^m,\quad \delta_{52}=[2m(m+1)+x^2k^2]I_m(kx)-2kxI_{m-1}(kx)\\
&\delta_{53}=2m(m+1)/k^m,\quad \delta_{54}=[2m(m+1)+x^2k^2]K_m(kx)+2kxK_{m-1}(kx)]\\
&\delta_{55}=-im\alpha,\quad \delta_{56}=\alpha[m^2+\beta k^2x^4]\\
&\delta_{61}=k^m[2m(m-1)+k^2x^2],\quad \delta_{62}=2m[kxI'_m(kx)-I_m(kx)]\\
&\delta_{63}=-[2m(m+1)+k^2x^2]/k^m\\
&\delta_{64}=2m[kxK'_m(kx)-K_m(kx)]\\
&\delta_{65}=-i\alpha\left[1+\frac{h^2}{12a^2k^2}(m^4-2m^2+1)+\beta k^2x^4\right]\\
&\delta_{66}=\alpha m.
\end{aligned} \quad (28)$$

It may be seen, that for a rigid structure the elements of the fifth and sixth line and column vanish. We then obtain the results previously derived by Bauer[3]. Frictionless liquid renders the remaining elements

$$\delta_{11} = s^2 + \frac{T_0}{\rho a^3} m(m^2-1), \quad \delta_{13} = \left[s^2 - \frac{T_0}{\rho a^3} m(m^2-1) \right], \quad \delta_{61} = sk^m, \quad \delta_{63} = -\frac{s}{k^m},$$

$$\delta_{65} = -i\bar{\alpha}\left\{1 + \frac{h^2}{12a^2k^2}(m^2-1)^2 + \bar{\beta}s^2\right\},$$

$$\delta_{66} = \bar{\alpha}m, \quad \delta_{31} = mk^m, \quad \delta_{33} = \frac{m}{k^m}, \quad \delta_{35} = -is,$$

where

$$\bar{\alpha} = \frac{\bar{D}}{\rho a^3} \quad \text{and} \quad \bar{\beta} \equiv \frac{\bar{\rho}(1-\bar{\nu}^2)b^2}{E}.$$

For a rigid structure we obtain the frequencies of the frictionless liquid (Fig. 2)

$$s^2 = -\frac{T_0 m(m^2-1)(1-k^{2m})}{\rho a^3(1+k^{2m})}$$

while for no liquid at all, the frequency[7] below is obtained. For an elastic cylinder without a liquid layer the determinant reduces to

$$\begin{vmatrix} \delta_{55} & \delta_{56} \\ \delta_{65} & \delta_{66} \end{vmatrix} = 0$$

which yields the frequencies

$$s^2 = -\frac{E}{2\bar{\rho}(1-\bar{\nu}^2)b^2}\left\{m^2+1+\frac{h^2}{12b^2}(m^2-1)^2 \mp \sqrt{m^4-2m^2+1-2(m^2-1)^3\frac{h^2}{12b^2}+\left(\frac{h^4}{144b^4}\right)(m^2-1)^4}\right\}$$

as was also shown for the simple shell without z-dependency by Reismann and Pawlik[8].

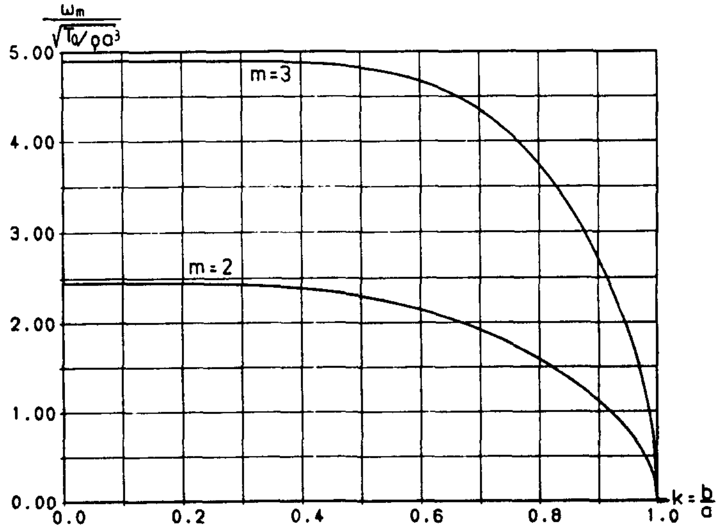

Fig. 2. Uncoupled liquid frequency for frictionless liquid.

2.2. *Viscous liquid inside an elastic shell*

If the viscous liquid is inside an elastic container $r = a$ with a free liquid surface at $r = b$, we have to employ the following for the determination of the frequency equation

$$\frac{\partial \xi}{\partial t} = u \quad \text{and} \quad \frac{\partial \eta}{\partial t} = v \qquad \text{at } r = a \tag{29}$$

as well as eqns (7) and (8) and (9) and (10) at $r = a$, where b has to be substituted by a. From eqns (9) and (10), we obtain eqn (21) in which T_0 has to be replaced by $-T_i$ and a by b. Equation (22) which stems from eqn (10) needs only an exchange of a by the value b. From the above eqns (29), eqns (23) and (24) are obtained, if in these, k is set equal to unity and b is replaced by a. Finally, the shell eqns give eqns (25) and (26), in which $k = 1$ and b has to be replaced by a. The elements of the frequency determinant are therefore given by:

$$\delta_{11} = k^4x^4 + 2k^2x^2m(m-1) - T_i^*km(m^2-1)$$

$$\delta_{12} = 2mk^2x^2[kxI_{m-1}(kx) - (m+1)I_m(kx)] - T_i^*km(m^2-1)I_m(kx)$$

$$\delta_{13} = -T_i^*km(m^2-1) - [2m(m+1)k^2x^2 + k^4x^4]$$

$$\delta_{14} = -2mk^2x^2[kxK_{m-1}(kx) + (m+1)K_m(kx)] - T_i^*km(m^2-1)K_m(kx)$$

$$\delta_{15} = \delta_{16} = 0$$

$$\delta_{21} = 2m(m-1)k^m, \quad \delta_{22} = [2m(m+1) + k^2x^2]I_m(kx) - 2kxI_{m-1}(kx)$$

$$\delta_{23} = 2m(m+1)k^{-m}, \quad \delta_{24} = [2m(m+1) + k^2x^2]K_m(kx) + 2kxK_{m-1}(kx)$$

$$\delta_{25} = \delta_{26} = 0$$

$$\delta_{31} = m, \quad \delta_{32} = mI_m(x), \quad \delta_{33} = m, \quad \delta_{34} = K_m(x)$$

$$\delta_{35} = -ix^2, \quad \delta_{36} = 0$$

$$\delta_{41} = m, \quad \delta_{42} = xI'_m(x), \quad \delta_{43} = -m, \quad \delta_{44} = xK'_m(x), \quad \delta_{45} = 0, \quad \delta_{46} = -x^2$$

$$\delta_{51} = 2m(m-1), \quad \delta_{52} = [2m(m+1) + x^2]I_m(x) - 2xI_{m-1}(x)$$

$$\delta_{53} = 2m(m+1), \quad \delta_{54} = [2m(m+1) + x^2]K_m(x) + 2xK_{m-1}(x)$$

$$\delta_{55} = -im\alpha, \quad \delta_{56} = \alpha[m^2 + \beta x^4]$$

$$\delta_{61} = -[2m(m-1) + x^2], \quad \delta_{62} = -2m[xI'_m(x) - I_m(x)], \quad \delta_{63} = [2m(m+1) + x^2]$$

$$\delta_{64} = -2m[xK'_m(x) - K_m(x)], \quad \delta_{65} = -i\alpha\left[1 + \frac{h^2}{12a^2}(m^4 - 2m^2 + 1) + \beta x^4\right]$$

$$\delta_{66} = \alpha m.$$

With these results the numerical values of the coupled frequencies may be obtained.

3. NUMERICAL EVALUATIONS AND CONCLUSIONS

For a liquid column around an elastic center shell the frequency eqn (27), with the elements (28) has been evaluated numerically for the various parameters of surface tension $T_0a/\rho v^2$, of density ratio $\bar{\rho}/\rho$, shell wall thickness ratio h/a, diameter ratio $k = b/a$ and stiffness parameter $\gamma = \bar{\rho}(1-\bar{v}^2)v^2k^2/Ea^2$. The results are given as real (——) and imaginary parts (---) of $S = (\bar{\sigma}a^2/v) + (i\bar{\omega}a^2/v)$ as a function of the ratio of the shell diameter to the liquid diameter $k = b/a$. This means, that if k is in the vicinity of $k = 1$, the liquid column is a very thin one, while with decreasing magnitude of k, the annular liquid column becomes thicker. We notice first of all, that for a viscous liquid system more than one root appears for each mode m. The numerical evaluation has been restricted to the mode $m = 2$, for which the natural frequency for a frictionless liquid around a rigid center shell may be obtained from eqn (30) (Fig. 2). This root is represented as S_0 in the following figures. The natural frequencies of the non-viscous liquid are proportional to the square root of the surface tension T_0, proportional to the inverse of the square root of the density of the liquid are exhibited in Fig. 2 as function of the diameter ratio $k = b/a$. It decreases with increasing diameter ratio $k = b/a$. The uncoupled natural frequencies for $m = 2$ and viscous liquid around a rigid center shell is marked with $\gamma = 0$ (Fig. 3a), which expresses nothing but the limit case of the modulus of elasticity $E \to \infty$. The uncoupled frequencies of the elastic shell with no liquid around it are shown in Fig. 4a as horizontal lines (–·–·–), exhibiting only an imaginary part. We are now interested in the coupled frequencies of the system, which yield for each mode, two coupled liquid frequencies (belonging to the oscillatory natural liquid frequency) and four coupled structural frequencies. In addition, there appear for the viscous liquid around a rigid center shell, an infinite number of pure real roots in each mode, which are describing a fast aperiodic motion. In the case of an elastically vibrating center shell, these roots remain real and render an aperiodic motion (see Fig. 5).

The coupled liquid roots are presented in Fig. 3a for the density ratio $\bar{\rho}/\rho = 2$, i.e. a shell density $\bar{\rho}$ of twice the density ρ of the liquid, for a thickness ratio of the shell of $h = 0.01a$ and a tension parameter $T_0 a/\rho v^2 = 1000$. The mode shown here is that of $m = 2$, which means the vibration is in an elliptic geometry. The parameter γ varies in the range of $10^{-9} \leqslant \gamma \leqslant 10^{-4}$.

The roots for a non-viscous liquid around a rigid shell are given by S_0, which is a pure conjugate imaginary root. It shows for increasing diameter ratio $k = b/a$, first a slight decrease of frequency, for larger k values ($k < 1$) then rapidly decreases to zero. The curve marked with $\gamma = 0$ is the oscillatory root of the viscous liquid for a rigid center shell, since $\gamma \to 0$ means, that the elasticity parameter $(Ea^2/\bar{\rho}(1-\bar{v}^2)v^2k^2) \to \infty$. This renders the roots as obtained already by Bauer[3]. It exhibits a damped oscillatory root with a decay magnitude shown for the real part (——) marked with $\gamma = 0$. With decreasing liquid thickness the decay magnitude increases, while the oscillatory part (– – –), i.e. the frequency, decreases. At about $k = b/a \approx 0.83$, the frequency of this damped oscillation becomes zero, indicating that in the range of $0.83 < k < 1$, i.e. for the smaller liquid thicknesses, the liquid layer is no longer able to oscillate, but merely performs an aperiodic motion, which with increasing k-magnitude exhibits less decay. For thin liquid layers therefore, the aperiodic motion of the liquid persists for a long time, but it does not oscillate. For a rather stiff shell

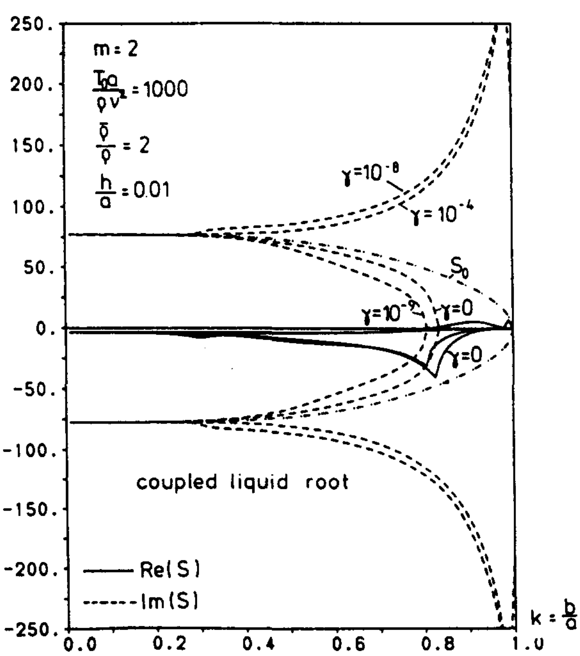

Fig. 3(a). Coupled and uncoupled frequencies of liquid and shell around center shell (–·–·– uncoupled frictionless liquid frequency, –··–··– uncoupled structural frequencies).

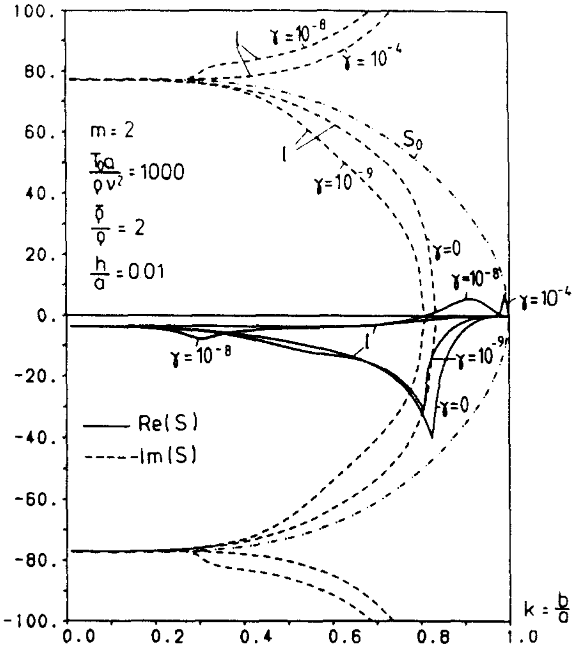

Fig. 3(b).

of $\gamma = 10^{-9}$ the coupled roots of the liquid–shell system show similar behavior as those of the rigid shell in the lower k range. Above $k \approx 0.79$ the decay decreases again and renders again an aperiodic motion with decreasing liquid layer thickness. For larger γ values, meaning more elastic shell behavior, the roots exhibit increasing decay magnitude and increasing frequency with increasing k-ratio. This may be seen better in Fig. 3b for $\gamma = 10^{-8}$ and 10^{-4}.

It may be noticed, that for $\gamma = 10^{-8}$ the coupled liquid frequency increases with decreasing layer thickness and that at $k \gtrsim 0.8$ the liquid motion becomes unstable. Similar results are obtained for an even less stiff shell ($\gamma = 10^{-4}$), where with increasing k value.

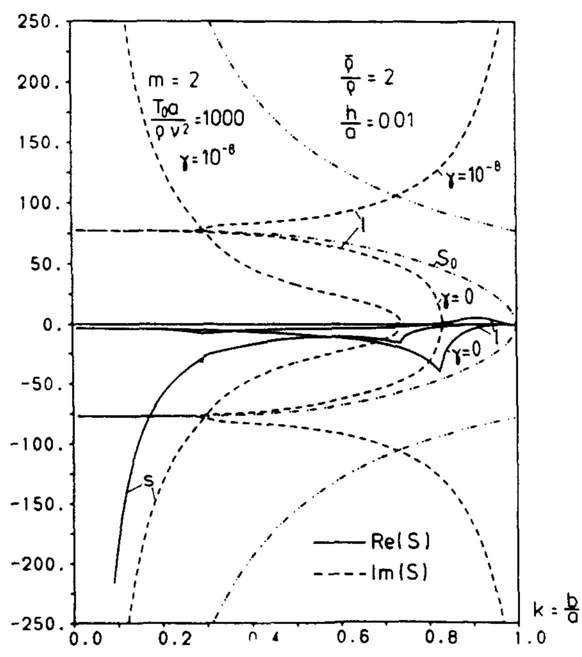

Fig. 3(c).

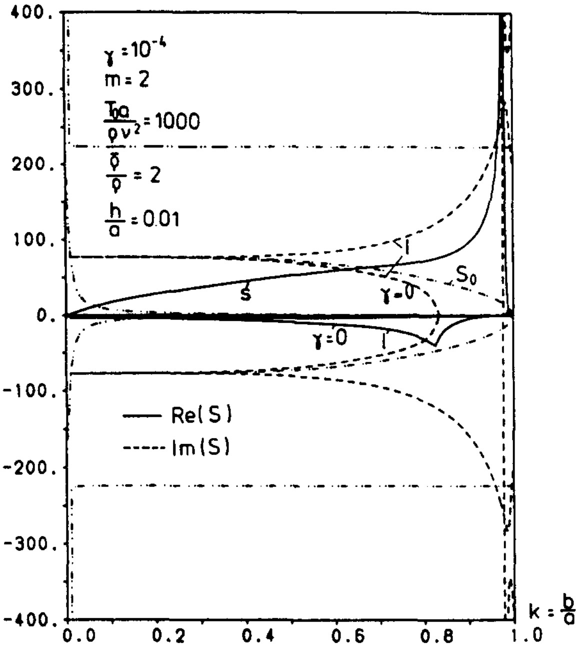

Fig. 4(a). Coupled and uncoupled frequencies of liquid and shell around center shell (–·–·– uncoupled frictionless liquid frequency, –··–··– uncoupled structural frequencies.

the oscillatory frequency increases, while at $k \gtrapprox 0.97$ the liquid becomes unstable. Figure 3c exhibits the real and imaginary part of S for $\gamma = 10^{-8}$ as a function of the diameter ratio k. The values S_0 of the frictionless liquid (–·–·–) and the damped frequency for the viscous liquid around a rigid center shell ($\gamma = 0$) are shown again for convenience. In addition the coupled structural frequencies are presented. The –··–··– line represents the uncoupled structural frequency, which decreases with increasing shell radius b. Again we notice that the coupled liquid frequency increases with increasing k, while its decay decreases slightly until at $k \gtrapprox 0.8$ the real part of this root becomes positive, indicating an oscillation of high frequency with increasing amplitude, i.e. oscillatory instability. The coupled structural root

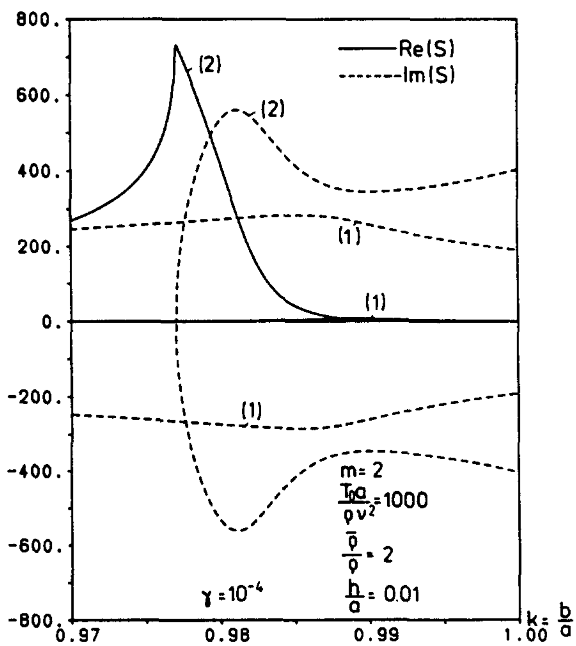

Fig. 4(b).

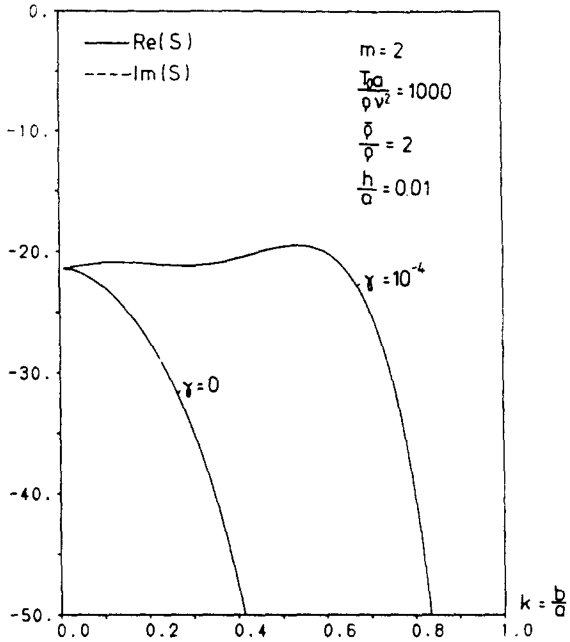

Fig. 5. Additional root in case of viscous liquid.

exhibits for a thick liquid layer, a very strong decay at a high oscillation frequency. With decreasing thickness of the liquid layer the decay magnitude decreases rapidly, indicating a longer oscillatory decay motion with a much smaller frequency. It may be noticed that the oscillation frequency of the coupled structural motion is much lower than that of the uncoupled shell motion. Finally at about $k \approx 0.73$ the shell ceases to oscillate and performs for $k > 0.73$, an aperiodically damped motion. The second structural frequency is very large and not in the range of the numerical values presented in Fig. 3c.

In Figs 4a and b we present the coupled roots again for an elasticity parameter $\gamma = 10^{-4}$. The uncoupled structural frequencies are both shown by the –···–··– lines. The liquid frequencies S_0 for frictionless liquid and that for a viscous liquid around a rigid shell ($\gamma = 0$) are again presented for convenience. It may be noticed that the structural frequency is unstable for all k values and that it is a diverging instability along nearly all the range of k. In Fig. 4b the range of $0.97 < k < 1$ is also shown. It is seen that the coupled structural frequency [marked as (2)] is divergingly unstable and becomes oscillatorily unstable above $k \approx 0.977$. The coupled liquid root is shown and marked as (1). We detect the previously mentioned instability and a strong decrease of the frequency. As in the case of a rigid shell, other roots appear for $m = 2$, which are strongly decaying. In Fig. 5 one of these roots is exhibited for the rigid shell ($\gamma = 0$) and for $\gamma = 10^{-4}$. We observe here that the additional root is again a fast decaying aperiodic motion, but that for an elastic shell it remains in a part of the k range nearly of the same magnitude. Finally the coupled liquid root is presented in the S plane with the elasticity and tension parameters as variables. It is seen, that increasing liquid surface tension parameter increases the oscillation frequency and also slightly increases the decay magnitude (Fig. 6). For decreasing shell stiffness (γ) the frequency of the coupled liquid root is hardly changed, while the decay magnitude strongly increases. This means, that with a stiffer shell the liquid oscillation exhibits stronger damping. Finally we determine the location, at which the liquid layer around a rigid center-core ceases to oscillate (Fig. 7). It may be noticed that with decreasing surface tension parameter $T_0 a/\rho v^2$ the liquid layer, for which aperiodic motion occurs, is becoming thicker. For a thick liquid layer with the surface tension parameter $T_0 a/\rho v^2 = 4$, the liquid surface oscillates in a damped fashion only for $k < 0.3$, while for liquid layers with a thickness of the magnitude smaller than $0.7a$ the liquid is not capable of oscillating any longer, but just performs an aperiodic motion if disturbed.

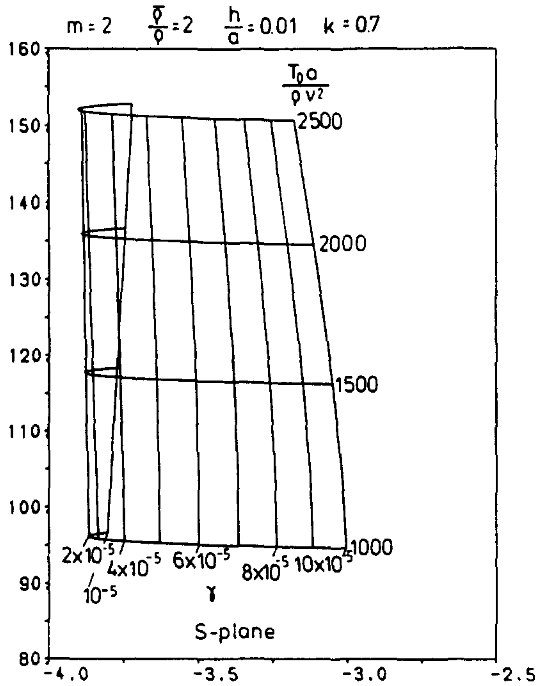

Fig. 6. Coupled liquid frequency in S plane.

In conclusion, we may state:

(a) that for a less stiff shell ($\gamma = 10^{-4}$) the coupled liquid root becomes unstable for very thin liquid layers, $k \gtrsim 0.97$;
(b) that for a stiffer shell, i.e. larger modulus of elasticity ($\gamma = 10^{-8}$), the coupled liquid root exhibits an instability for thin layers ($k > 0.8$) and shows a stronger instability for thinner layers;
(c) that for an even further increase of E ($\gamma = 10^{-9}$) the coupled liquid root has a small instability at $k \approx 1$;

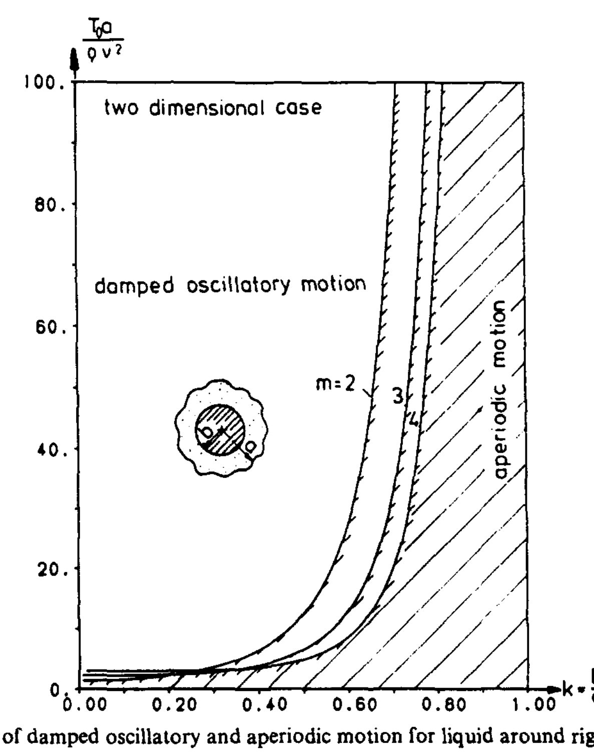

Fig. 7. Area of damped oscillatory and aperiodic motion for liquid around rigid center-core.

(d) that for a less stiff shell ($\gamma = 10^{-4}$) the structural root for the radial direction becomes strongly unstable, exhibiting diverging instability until $k = 0.977$ and for even thinner liquid layers ($k > 0.977$) exhibiting an oscillatory instability.

The physical significance of the obtained results here calls for some discussion. The main interest appears in that we observe a case where viscosity provides instability, whereas an ideal liquid always renders stability. Small perturbations of the structure lead to perturbations of the liquid surface and vice versa, where in the case of a viscous liquid (for which different phase relations appear) the flow field of the liquid is quite different to ideal liquids, which exhibit slipping in an angular direction. These perturbations result in liquid surface displacements, thus changing the surface tension restoring force. In the case of a viscous liquid, the additional disturbance in the angular direction by the motion of the elastic shell, is not adequate to dampen out the perturbation. A similar explanation of the instability of the coupled motion of the elastic shell may be stated. Figure 4b shows that the root belonging to the angular (η) motion becomes unstable and that the instability flips from a diverging to an oscillatory instability for thin liquid layers, where the oscillatory interaction of the liquid motion is more pronounced. These effects (phase shifts and additional excitation of the liquid through the angular motion of the elastic shell and vice versa), which are not present for frictionless liquid seem to be responsible for such instabilities.

REFERENCES

1. H. F. Bauer, Coupled oscillations of a solidly rotating liquid bridge. *Acta Astr.* **9**, 547–563 (1982).
2. H. F. Bauer, Natural damped frequencies of an infinitely long column of immiscible viscous liquids. *Z. Angew. Math. Mech.* **64**, 475–490 (1984).
3. H. F. Bauer, Natural damped frequencies of an infinitely long column of immiscible viscous liquid. *Forsch. Geb. IngWes.* **49**, 117–126 (1983).
4. Lord Rayleigh, On the instability of cylindrical fluid surfaces. *Phil. Mag.* **34**, 177–180 (1892).
5. H. Lamb, *Hydrodynamics*, pp. 471–473. Dover, New York (1945).
6. J. A. F. Plateau, Experimental and theoretical researches on the figures of equilibrium of a liquid mass withdrawn from the action of gravity. Smithunion Institute Annual Reports: 1863, pp. 207–285; 1864, pp. 286–369; 1865, pp. 411–435; 1866, pp. 255–289. Government Printing Office, Washington.
7. A. W. Leissa, *Vibration of Shells*, National Aeronautic and Space Administration, NASA-SP-288 (1973).
8. H. Reismann and P. S. Pawlik, Plane-strain dynamic response of a cylindrical shell: a comparison study of three different shell theories. *J. appl. Mech.* **35**, 297–305 (1968).